

A projection algorithm for non-monotone variational inequalities

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Abstract

We introduce and study the convergence properties of a projection-type algorithm for solving the variational inequality problem for point-to-set operators. No monotonicity assumption is used in our analysis. The operator defining the problem is only assumed to be continuous in the point-to-set sense, i.e., inner- and outer-semicontinuous. Additionally, we assume non-emptiness of the so-called dual solution set. We prove that the whole sequence of iterates converges to a solution of the variational inequality. Moreover, we provide numerical experiments illustrating the behavior of our iterates. Through several examples, we provide a comparison with a recent similar algorithm.

Keywords: Variational inequality, Projection algorithms, outer-semicontinuous operator, inner-semicontinuous operator.

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1 Introduction

Variational inequalities were introduced in 1966 by Hartman and Stampacchia (see [18]), and have numerous important applications in physics, engineering, economics, and optimization theory (see, e.g., [14, 18, 21, 22] and the references therein). The variational inequality problem for a point-to-set operator $T : \text{dom}(T) \subseteq \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and a nonempty closed and convex set $C \subset \text{dom}(T)$, is stated as

$$\text{Find } x_* \in C \text{ such that } \exists u_* \in T(x_*), \text{ with } \langle u_*, x - x_* \rangle \geq 0, \quad \forall x \in C. \quad (1)$$

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We will denote by S_* the solution set of the Problem (1). This problem can be studied via its so-called *dual formulation*, which is stated as

$$\text{Find } x_* \in C \text{ such that } \forall x \in C, \text{ and } \forall u \in T(x), \quad \langle u, x - x_* \rangle \geq 0. \quad (2)$$

We denote the solution set of Problem (2) by S_0 . It is easy to see that S_0 is a closed and convex set. However, in general, S_* is not. Most of the available convergence analysis for variational inequalities rely on some kind of monotonicity assumption. Namely, when T is point-to-set, i.e., $T(x)$ is a subset of \mathbb{R}^n , a standard assumption for analyzing Problem (1) is either maximal monotonicity (see e.g., [1, 30]), pseudo-monotonicity (see e.g., [5, 11]) or quasi-monotonicity [31]. In the point-to-point case, continuity of T as well as $S_* \neq \emptyset$ are standard assumptions for analyzing (1), (see e.g. [4, 26, 27, 32]). In view of its wide range of applications it is imperative to consider general versions of (1) which relax the standard assumptions mentioned above.

For solving variational inequalities, projection-type methods (see, e.g., [3, 12, 13, 29]) are very popular because the iteration can be performed cheaply when the set C has a simple structure (e.g., when C is a ball or a polyhedral). The other methods of choice for variational inequalities can be cast as proximal-like or interior point methods (see e.g., [7, 9, 10, 15, 28, 31]). The former ones, however, may result in iterations which are as complex as solving the original problem and usually involve some kind of monotonicity assumption on T . A main drawback of the latter methods (Interior Point Methods) is “the perceived lack of an efficient warmstarting scheme which would enable the use of information from a previous solution of a similar problem” (quote taken from [16]).

Namely, in the present paper we devise a projection-type method for point-to-set variational inequalities, and establish convergence to a solution of Problem (1) under two basic assumptions: (i) non emptiness of the set S_0 , and (ii) a suitable concept of continuity for point-to-set operators. The concept of continuity we use in (ii) can be found, e.g., in [8], and is formally stated in Definition 2.1.

If T is point-to-point and maximally monotone, then it will automatically satisfy assumption (ii), as well as (i) whenever $S_* \neq \emptyset$. Hence our analysis is valid for these cases when the problem has a solution.

It will be proved in Proposition 2.2 that, when C is contained in the domain of T , assumption (ii) implies $S_0 \subseteq S_*$. So, the existence of solutions of (2) implies $S_* \neq \emptyset$. For the inclusion $S_* \subseteq S_0$ to hold, an extra condition, such as pseudo monotonicity, is needed (see [24, Lemma 1]).

Assumption (i) has been used in [29] for variational inequalities with a point-to-point operator, and it has also been used in [6, 20] for the equilibrium problem (i.e., for the point-to-point case). As far as we know, assumptions (i) and (ii) haven't been used for the point-to-set case. Condition (i) together with $S_0 = S_*$ is a well-known example of an assumption that does not involve a monotonicity requirement on T , see e.g., [4, 25, 34], where this assumption is used in the point-to-point case.

The algorithm considered in [29] uses assumption (i) for the point-to-point case. The

difference between their method and ours can be explained as follows. In [29], the current point x^k is projected onto a subset that contains the solution set. At each iteration of our algorithm, we project the same point x^0 onto a set which is strictly smaller than the one used in [29]. The way of defining the iterates in [29] allows for the use of Fejér convergence in the analysis, which is a classical tool for this kind of projection algorithms. Because we do not project the current iterate, but a fixed point x^0 instead, our convergence analysis cannot make use of such a powerful tool. Moreover, if our sequence does not have finite termination, we can characterize the limit as the closest point to the initial iterate x^0 in the set $\bar{co}(\tilde{S}_*)$ (see Proposition 4.5 and Theorem 4.9). Namely, $x^k \rightarrow \bar{x}$ where $\bar{x} = P_{\bar{co}(\tilde{S}_*)}(x^0)$. Unlike [29], our method can be used for point-to-set monotone, pseudo- or quasi-monotone variational inequalities, such as those in [1, 5, 11, 31].

The paper is organized as follows. In section 2 we give some notation, definitions, and useful results. In Section 3 we define the linesearch and the algorithm. In Section 4 we provide the convergence analysis of the algorithm. Section 5 presents numerical examples and comparisons. Finally, Section 6 contains our conclusions and open problems.

2 Preliminaries

In this section we introduce some known definitions, facts and properties that will be used in the sequel. First, we fix the notation and recall some definitions. The inner product in \mathbb{R}^n is denoted by $\langle \cdot, \cdot \rangle$ and its norm by $\| \cdot \|$. For C a nonempty, convex and closed subset of \mathbb{R}^n , the *orthogonal projection of x onto C* is denoted by $P_C(x)$, and defined as the unique point in C such that $\|P_C(x) - x\| \leq \|y - x\|$ for all $y \in C$. Let $(x^k)_{k \in \mathbb{N}}$ be a sequence in \mathbb{R}^n , we denote the set of its cluster points by $Cl(x^k)_{k \in \mathbb{N}}$. For the point-to-set operator T , we define the *domain of T* as $\text{dom}(T) := \{x \in \mathbb{R}^n : T(x) \neq \emptyset\}$, and the *graph of T* as $Gr(T) := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^n : u \in T(x)\}$.

We begin with a concept of continuity for a point-to-set operator. Our definitions are standard and taken from [8].

Definition 2.1 *Let $T : \text{dom}(T) \subset \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a point-to-set mapping. Then,*

- (a) *T is said to be outer-semicontinuous (OSC), if and only if, the graph of T is closed.*
- (b) *T is said to be inner-semicontinuous (ISC) at $x \in \text{dom}(T)$, if and only if, for any $y \in T(x)$ and for any sequence $(x^k)_{k \in \mathbb{N}} \subset \text{dom}(T)$ such that $x^k \rightarrow x$ there exist a sequence $(y^k)_{k \in \mathbb{N}}$ such that $y^k \in T(x^k)$ for all $k \in \mathbb{N}$ and $y^k \rightarrow y$.*
- (c) *T is said to be upper-semicontinuous (USC) for all $x \in \text{dom}(T)$, if and only if, for all open $W \subset \mathbb{R}^n$ such that $W \supset T(x)$ there exists a neighborhood U of x such that $T(x') \subset W$ for all $x' \in U$.*
- (d) *T is said to be continuous if it is ISC and OSC.*

An important proposition that relates the sets S_0 and S_* is as follows.

Proposition 2.2 *Let the point-to-set mapping $T : \text{dom}(T) \subset \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be ISC in all $x \in \text{dom}(T)$, then $S_0 \subseteq S_*$.*

Proof. Take $x^* \in S_0 \cap \text{dom}(T)$, then for all $(y, v) \in \text{Gr}(T)$ such that $y \in C \cap \text{dom}(T)$ we have $\langle v, y - x^* \rangle \geq 0$. Now, for all $\alpha \in (0, 1)$ we have by convexity of C that $y_\alpha := (1 - \alpha)x^* + \alpha y \in C$ for all $y \in C \cap \text{dom}(T)$. Taking $u_* \in T(x^*)$, we have that there exist $v_*^\alpha \in T(y_\alpha)$ such that $\lim_{\alpha \rightarrow 0} v_*^\alpha = u_*$. Now, using that $x^* \in S_0$, we have that

$$0 \leq \langle v_*^\alpha, y_\alpha - x^* \rangle = \alpha \langle v_*^\alpha, y - x^* \rangle.$$

Dividing by $\alpha > 0$ and taking limits when α goes to zero, we get that $\langle u_*, y - x^* \rangle \geq 0$, for all $y \in C$, then $x^* \in S_*$. ■

Remark 2.3 The inclusion $S_0 \subseteq S_*$ has been established in [23,24,33]. These papers assume T to be *USC* and such that $T(x)$ is compact for all $x \in \text{dom}(T)$. More precisely, if T has closed images, then upper-semicontinuity implies outer-semicontinuity (see [8, Proposition 2.5.12 (b)(c)]) and hence our analysis includes the cases considered in [23,24,33]. As far as we know, Proposition 2.2 is new for T point-to-set and *ISC*. An example showing an operator T which is *OSC* and not *USC* can be found in [8, Example 2.5.8]. While *upper-semicontinuity* can be seen as a natural extension of the point-to-point continuity, it cannot express properly continuity of mappings in which $T(x)$ is unbounded (see, e.g., [8, Example 2.5.8]). Hence our choice of *OSC* over *USC*. In Example 5.4, we implement our algorithm for a point-to-set operator which is not *USC* but is continuous in the sense of Definition 2.1.

Now, we present some important facts on orthogonal projection that will be very useful for the well-definedness of the Linesearch presented in the Section 3.

Fact 2.4 Let $C \subseteq \mathbb{R}^n$ be a closed and convex set. For all $x, y \in \mathbb{R}^n$ and all $z \in C$ the following hold:

- (i) $\|P_C(x) - P_C(y)\|^2 \leq \|x - y\|^2 - \|(x - P_C(x)) - (y - P_C(y))\|^2$.
- (ii) $\langle x - P_C(x), z - P_C(x) \rangle \leq 0$.

Proof. See [36, Lemmas 1.1 and 1.2]. ■

2.1 Some useful results

The following three results are standard in the literature of variational inequalities, we add here their proof for convenience of the reader.

The next property will be used for the stopping criteria of the algorithm and in the finite termination of the **Linesearch F**.

Proposition 2.5 [13, Proposition 1.5.8] Given $T : \text{dom}(T) \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $C \subset \text{dom}(T) \subset \mathbb{R}^n$. If for some $u \in T(x)$ and $\beta > 0$, $x = P_C(x - \beta u)$, then $x \in S_*$.

Proof. Due to Fact 2.4(ii), we have $\langle x - \beta u - P_C(x - \beta u), y - P_C(x - \beta u) \rangle \leq 0$, for all $y \in C$ using that $x = P_C(x - \beta u)$ and that $\beta > 0$ follow $\langle u, y - x \rangle \geq 0$ for all $y \in C$. Proving that $x \in S_*$. ■

Now we show a lemma which ensures that the hyperplanes used in the algorithm contain the solution set of Problem (2).

Lemma 2.6 [4, Lemma 2.17] For any $z \in C$ and $u \in T(x)$, define $H(z, u) := \{y \in \mathbb{R}^n : \langle u, y - z \rangle \leq 0\}$. Then, $S_0 \subseteq H(z, u)$.

Proof. For $x_* \in S_0$ we have $\langle u, x - x_* \rangle \leq 0$ for all $(x, u) \in \text{Gr}(T)$ with $x \in C$, then $x_* \in H(z, u)$. ■

We now prove a lemma useful for proving that the hyperplanes used in the algorithm separate the current iterate and the solution set.

Lemma 2.7 Let $C \subset \mathbb{R}^n$ be a closed, convex and nonempty set. Take $x \in C$ and $z = P_C(x - \beta u)$, with $\beta > 0$ and $u \in \mathbb{R}^n$. Assume that

$$(i) \quad \bar{x} = \alpha z + (1 - \alpha), \text{ with } \alpha \in (0, 1).$$

$$(ii) \quad (\bar{x}, \bar{u}), (x, u) \in \text{Gr}(T).$$

$$(iii) \quad \langle \bar{u}, x - z \rangle \geq \delta \langle u, x - z \rangle.$$

Then, $x \in H(\bar{x}, \bar{u})$ implies that $x \in S_*$. With $H(x, u)$ as in Lemma 2.6.

Proof. Since $x \in H(\bar{x}, \bar{u})$ we have that $\langle \bar{u}, x - \bar{x} \rangle \leq 0$. Using Fact 2.4 (ii), we have

$$\begin{aligned} 0 &\geq \langle \bar{u}, x - \bar{x} \rangle = \alpha \langle \bar{u}, x - z \rangle \geq \alpha \delta \langle u, x - z \rangle \\ &= \frac{\alpha}{\beta} \langle z - (x - \beta u), x - z \rangle + \frac{\alpha}{\beta} \|x - z\|^2 \\ &\geq \frac{\alpha}{\beta} \delta \|x - z\|^2 \geq 0, \end{aligned} \tag{3}$$

implying that $x = z$. By Proposition 2.5, this implies that $x \in S_*$. ■

The next result will be used for proving the boundedness of the sequence generated by the algorithm and will play an important role for the convergence analysis presented in Section 4.

Lemma 2.8 [4, Lemma 2.10] Let S be a nonempty, closed and convex set. Let $x^0, x \in \mathbb{R}^n$. Assume that $x^0 \notin S$ and that $S \subseteq W(x) = \{y \in \mathbb{R}^n : \langle y - x, x^0 - x \rangle \leq 0\}$. Then, $x \in B[\frac{1}{2}(x^0 + \bar{x}), \frac{1}{2}\rho]$, where $\bar{x} = P_S(x^0)$ and $\rho = \text{dist}(x^0, S) = \|x^0 - P_S(x^0)\|$.

Proof. Since S is convex and closed, $\bar{x} = P_S(x^0)$ and $\rho = \text{dist}(x^0, S)$ are well-defined. $S \subseteq W(x)$ implies that $\bar{x} = P_S(x^0) \in W(x)$. Define $v := \frac{1}{2}(x_0 + \bar{x})$ and $r := x^0 - v = \frac{1}{2}(x^0 - \bar{x})$, then $\bar{x} - v = -r$ and $\|r\| = \frac{1}{2}\|x^0 - \bar{x}\| = \frac{1}{2}\rho$. Since $\bar{x} \in W(x)$, we can write

$$\begin{aligned} 0 &\geq \langle \bar{x} - x, x^0 - x \rangle = \langle \bar{x} - v + v - x, x^0 - v + v - x \rangle \\ &= \langle -r + (v - x), r + (v - x) \rangle = \|v - x\|^2 - \|r\|^2. \end{aligned}$$

Then, the result follows. ■

The following proposition will be used to show that the distance between consecutive iterates tends to zero. It is very well-know, but hard to track down, hence we include its proof here.

Proposition 2.9 *Let $x^0, x \in \mathbb{R}^n$ and define $W(x) := \{y \in \mathbb{R}^n : \langle y - x, x^0 - x \rangle \leq 0\}$, then it holds that $x = P_{W(x)}(x^0)$.*

Proof. Since, $x \in W(x)$ and $P_{W(x)}(x^0) \in W(x)$ using Proposition 2.4 (ii) we have,

$$\langle P_{W(x)}(x^0) - x, x^0 - x \rangle \leq 0 \quad (4)$$

$$\langle P_{W(x)}(x^0) - x, P_{W(x)}(x^0) - x^0 \rangle \leq 0. \quad (5)$$

Summing (4) and (5) we have that $\|P_{W(x)}(x^0) - x\|^2 \leq 0$, then $x = P_{W(x)}(x^0)$. ■

3 The linesearch and the algorithm

Our linesearch is a modification of a search strategy first introduced in 1997, see [19]. The authors of [19] use the square of the norm in the right-hand side of the inequality in the **Linesearch F** (**F** stands for *feasible* direction method). Later on, Konnov in [25] uses a linesearch as the one we use below, but for point-to-point mappings. Both [19] and [25] use the assumption $S_0 = S_*$.

Linesearch F (feasible direction) **Input:** $x \in C$, $\beta > 0$ and $\delta \in (0, 1)$.

Set $\alpha \leftarrow 1$ and $\theta \in (0, 1)$. Define $z = P_C(x - \beta u)$ with $u \in T(x)$

If $\forall u_\alpha \in T(\alpha z + (1 - \alpha)x)$, $\langle u_\alpha, x - z \rangle < \delta \langle u, x - z \rangle$ **then** $\alpha \leftarrow \theta\alpha$, **Else** Return α .

Output: (α) .

As mentioned in the Introduction, we will use in our analysis the following assumptions on T :

(A1) The feasible set C is contained on the domain of T , i.e., $C \subset \text{dom}(T)$.

(A2) T continuous on C , in the sense of the Definition 2.1(d).

(A3) The solution set S_0 of the Dual Problem (2) is not empty.

The fact that the **Linesearch F** has finite termination (and hence, well defined) is proved next.

Lemma 3.1 *Assume that (A1) holds and T is ISC at every point of C . If $x \in C$ and $x \notin S_*$, then **Linesearch F** stops after finitely many steps.*

Proof. Since T is ISC at x , given $u \in T(x)$ and $y_\alpha \rightarrow x$, with $y_\alpha = \alpha z + (1 - \alpha)x$ and $\alpha \in (0, 1)$ there exist $v_\alpha \in T(y_\alpha) : v_\alpha \rightarrow u$ when $\alpha \rightarrow 0$. Now, suppose that **Linesearch F** never stops, then we have:

$$\langle v_\alpha, x - z \rangle < \delta \langle u, x - z \rangle. \quad (6)$$

Taking limits in (6) when $\alpha \rightarrow 0$, we have

$$\langle u, x - z \rangle \leq \delta \langle u, x - z \rangle \Leftrightarrow (1 - \delta) \langle u, x - z \rangle \leq 0.$$

Since $\delta \in (0, 1)$, we have

$$0 \geq \langle u, x - z \rangle = \frac{1}{\beta} (\|x - z\|^2 + \langle z - (x - \beta u), x - z \rangle),$$

using Fact 2.4 (ii) we get $\|x - z\|^2 \leq \langle (x - \beta u) - z, x - z \rangle \leq 0$, which implies that $x = z$. Hence, $x \in S_*$ by Proposition 2.5. This contradicts our assumption $x \notin S_*$. Then, the well definition of **Linesearch F** follows. \blacksquare

Remark 3.2 The implementation of the **Linesearch F** for point-to-set mappings might be a nontrivial task. In Example 5.4 we present an operator T , for which this implementation is possible.

Recall from Section 2 that

$$H(z, v) := \{y \in \mathbb{R}^n : \langle v, y - z \rangle \leq 0\} \quad (7)$$

and

$$W(x) := \{y \in \mathbb{R}^n : \langle y - x, x^0 - x \rangle \leq 0\}. \quad (8)$$

These halfspaces (as well as its intersections) have been widely used in the literature, e.g., [2, 4, 6, 29, 35].

Now we describe the Algorithm.

Algorithm F (Feasible direction algorithm) Given $(\beta_k)_{k \in \mathbb{N}} \subset [\check{\beta}, \hat{\beta}]$ such that $0 < \check{\beta} \leq \hat{\beta} < +\infty$ and $\delta \in (0, 1)$.

Initialization: Take $x^0 \in C$, define $\tilde{H}_0 := \mathbb{R}^n$ and set $k \leftarrow 0$.

Step 1: Set $z^k = P_C(x^k - \beta_k u^k)$ with $u^k \in T(x^k)$ and

$$\alpha_k = \text{Linesearch } F(x^k, \beta_k, \delta), \quad (9)$$

i.e., (α_k, z^k) satisfy

$$\{ \langle \bar{u}^k, x^k - z^k \rangle \geq \delta \langle u^k, x^k - z^k \rangle \}. \quad (10)$$

with $\bar{u}^k \in T(\alpha_k z^k + (1 - \alpha_k)x^k)$.

Step 2 (Stopping Criterion): If $z^k = x^k$ or $z^k = P_C(z^k - v^k)$ with $v^k \in T(z^k)$, then stop. Otherwise,

Step 3: Set

$$\bar{x}^k := \alpha_k z^k + (1 - \alpha_k)x^k, \quad (11a)$$

$$\tilde{H}_k := \tilde{H}_{k-1} \cap H(\bar{x}^k, \bar{u}^k), \quad (11b)$$

$$\text{and } x^{k+1} := P_{C \cap \tilde{H}_k \cap W(x^k)}(x^0); \quad (11c)$$

Step 4: If $x^{k+1} = x^k$, then stop. Otherwise, set $k \leftarrow k + 1$ and go to **Step 1**.

4 Convergence Analysis

Our goal in this section is to establish the convergence of the algorithm. First of all, let us see that the stopping criterion is well defined.

Proposition 4.1 *If the Algorithm F stops at Step 2, then x^k or z^k are solutions.*

Proof. It is a direct consequence of the definition of z^k , v^k and Proposition 2.5. ■

Proposition 4.2 *Let $(\bar{x}^k)_{k \in \mathbb{N}}$, $(x^k)_{k \in \mathbb{N}}$ and $(\bar{u}^k)_{k \in \mathbb{N}}$ be sequences generated by Algorithm F. If $x^k \in H(\bar{x}^k, \bar{u}^k)$ (see (7)), then $x^k \in S_*$.*

Proof. Follows applying Lemma 2.7 for $\alpha = \alpha_k$, $x = x^k$, $\bar{x} = \bar{x}^k$, $u = u^k$ and $\bar{u} = \bar{u}^k$, using **Linesearch F**. ■

The following remark points to a useful algebraic property of the sequence generated by **Algorithm F**. This property is a direct consequence of the **Linesearch F**.

Remark 4.3 Let $(x^k)_{k \in \mathbb{N}}$ and $(\alpha_k)_{k \in \mathbb{N}}$ be sequences generated by **Algorithm F**, using (3), we get

$$\forall k \in \mathbb{N}: \quad \langle \bar{u}^k, x^k - \bar{x}^k \rangle \geq \frac{\alpha_k}{\hat{\beta}} \delta \|x^k - z^k\|^2. \quad (12)$$

Proposition 4.4 If $x^{k+1} = x^k$, then $x^k \in S_*$.

Proof. If $x^{k+1} = P_{C \cap \tilde{H}_k \cap W(x^k)}(x^0) = x^k$, then $x^k \in \tilde{H}_k$, which implies that $x^k \in H(\bar{x}^k, \bar{u}^k)$, by Proposition 4.2 we have that $x^k \in S_*$. ■

If **Algorithm F** stops in a finite number of iterations, then by Propositions 4.1 and 4.4 the last iterate is a solution. Hence, it is enough to establish convergence when the algorithm does not stop. Therefore, from now on, we suppose that the sequence $(x^k)_{k \in \mathbb{N}}$ generated by the **Algorithm F**, is infinite and $x^k \notin S_*$ for all $k \in \mathbb{N}$. The next result shows that the projection step is well-defined.

Proposition 4.5 Let \tilde{H}_k be as in (11b), and define $\tilde{S}_* := \bigcap_{k \in \mathbb{N}} \tilde{H}_k \cap S_*$. Then, $\tilde{S}_* \subset H(\bar{x}^k, \bar{u}^k) \cap W(x^k)$ for all $k \in \mathbb{N}$ and $\tilde{S}_* \neq \emptyset$.

Proof. By definition we have that $\tilde{S}_* \subset C \cap H(\bar{x}^k, \bar{x}^k)$ for all $k \in \mathbb{N}$. Now we prove by induction that $\tilde{S}_* \subset W(x^k)$ for all $k \in \mathbb{N}$. For $k = 0$ we have that $\tilde{S}_* \subset W(x^0) = \mathbb{R}^n$, suppose that $\tilde{S}_* \subset W(x^k)$, then by Fact 2.4 (ii), we obtain $\langle x_* - x^{k+1}, x^0 - x^{k+1} \rangle \leq 0$, for all $x_* \in \tilde{S}_*$. This implies $x_* \in W(x^{k+1})$. Then, the result follow by induction. By Lemma 2.6 we have that $S_0 \subset H(\bar{x}^k, \bar{u}^k)$ for all $k \in \mathbb{N}$. By Assumption (A2) and Proposition 2.2 we deduce that $S_0 \subseteq S_*$, hence $S_0 \subseteq \tilde{S}_*$ and by Assumption (A3), $\tilde{S}_* \neq \emptyset$. ■

Now we prove the well definition of the iterates of **Algorithm F**.

Proposition 4.6 The sequence $(x^k)_{k \in \mathbb{N}}$ is well defined and $(x^k)_{k \in \mathbb{N}} \subset C$.

Proof. By definition of the solution set, we have that $S_* \subset C$, then by Proposition 4.5, for all $k \in \mathbb{N}$ the set $C \cap H(\bar{x}^k, \bar{u}^k) \cap W(x^k) \neq \emptyset$, closed and convex (note that C , $H(\bar{x}^k, \bar{u}^k)$ and $W(x^k)$ are convex and closed sets). Therefore, the projection step is well-defined. The fact that $x^k \in C$ for all $k \in \mathbb{N}$ follows from the definition of the iterates in (11c) and the fact that $x^0 \in C$. ■

The next result proves the boundedness of the sequence generated by the algorithm.

Proposition 4.7 The sequence generated by the algorithm satisfies that $(x^k)_{k \in \mathbb{N}} \subset B[\frac{1}{2}(x^0 + \bar{x}), \frac{\rho}{2}]$, where $\bar{x} := P_{S_0}(x^0)$ and $\rho = \|x^0 - P_{S_0}(x^0)\|$. Therefore, the sequence $(x^k)_{k \in \mathbb{N}}$ is bounded.

Proof. Since S_0 is a nonempty, convex and closed set and $x^0 \notin S_0$, we are in the hypothesis of Lemma 2.8. Using this lemma with $S = S_0$ and $x = x^k$, the result follows. ■

Next we show that the distance between consecutive iterates tends to zero.

Proposition 4.8 *The sequence $(x^k)_{k \in \mathbb{N}}$ satisfies that $\sum_{k=0}^{\infty} \|x^{k+1} - x^k\|^2 < \infty$, hence $\lim_{k \rightarrow \infty} \|x^{k+1} - x^k\| = 0$.*

Proof. By Proposition 2.9, for $x = x^k$ we have that $x^k = P_{W(x^k)}(x^0)$. Since $x^{k+1} \in W(x^k)$ then, by Fact 2.4 (i), we obtain that $0 \leq \|x^{k+1} - x^k\|^2 \leq \|x^{k+1} - x^0\|^2 - \|x^k - x^0\|^2$. Summing this inequality from $k = 0$ to ∞ and using the boundedness of the sequence $(x^k)_{k \in \mathbb{N}}$, we obtain that $\sum_{k=0}^{\infty} \|x^{k+1} - x^k\|^2 < \infty$. Therefore, $\lim_{k \rightarrow \infty} \|x^{k+1} - x^k\| = 0$. ■

We present next a key convergence result for our algorithm.

Theorem 4.9 *Let $(x^k)_{k \in \mathbb{N}}$ be the sequence generated by the **Algorithm F**. Then $Cl(x^k)_{k \in \mathbb{N}} \subseteq \tilde{S}_* \subseteq S_*$.*

Proof. We prove first that $Cl(x^k)_{k \in \mathbb{N}} \subseteq S_*$. Since $x^{k+1} \in H(\bar{x}^k, \bar{u}^k)$ for all $k \in \mathbb{N}$ then by definition of $H(\bar{x}, \bar{u})$ we have that $\langle \bar{u}^k, x^{k+1} - \bar{x}^k \rangle \leq 0$. Now,

$$0 \geq \langle \bar{u}^k, x^{k+1} - \bar{x}^k \rangle = \langle \bar{u}^k, x^{k+1} - x^k \rangle + \alpha_k \langle \bar{u}^k, x^k - z^k \rangle.$$

Using the same ideas as in (3) and Remark 4.3, we have that

$$\frac{\alpha_k \delta}{\tilde{\beta}} \|x^k - z^k\|^2 \leq \langle \bar{u}^k, x^k - x^{k+1} \rangle \leq \|\bar{u}^k\| \|x^k - x^{k+1}\|,$$

then passing to the limits for $k \rightarrow \infty$ we have by the boundedness of \bar{x}^k that,

$$\lim_{k \rightarrow \infty} \alpha_k \|x^k - z^k\| = 0. \quad (13)$$

Now we take a subsequence $(i_k)_{k \in \mathbb{N}}$ such that, $(\alpha_{i_k})_{k \in \mathbb{N}}$, $(\beta_{i_k})_{k \in \mathbb{N}}$, $(x^{i_k})_{k \in \mathbb{N}}$, $(u^{i_k})_{k \in \mathbb{N}}$ and $(z^{i_k})_{k \in \mathbb{N}}$ be convergent to $\tilde{\alpha}$, $\tilde{\beta}$, \tilde{x} , \tilde{u} and \tilde{z} respectively. This is possible by the boundedness of all sequences involved, note that by Assumption **(A2)**, we have that $Gr(T)$ is closed, then we have that $\tilde{u} \in T(\tilde{x})$. Now we consider two cases:

Case 1: $\lim_{k \rightarrow \infty} \alpha_{i_k} = \tilde{\alpha} > 0$. Consequence of (13), $\lim_{k \rightarrow \infty} \|x^{i_k} - z^{i_k}\| = 0$. Using the continuity of the projection $\tilde{x} = \lim_{k \rightarrow \infty} x^{i_k} = \lim_{k \rightarrow \infty} z^{i_k} = P_C(\tilde{x} - \tilde{\beta}\tilde{u})$. Then, $\tilde{x} = P_C(\tilde{x} - \tilde{\beta}\tilde{u})$, and Proposition 2.5 implies that $\tilde{x} \in S_*$.

Case 2: $\lim_{k \rightarrow \infty} \alpha_{i_k} = \tilde{\alpha} = 0$. Define $\tilde{\alpha}_k = \frac{\alpha_{i_k}}{\theta}$. Then,

$$\lim_{k \rightarrow \infty} \tilde{\alpha}_{i_k} = 0. \quad (14)$$

Define $\tilde{y}^k := \tilde{\alpha}_k z^k + (1 - \tilde{\alpha}_k)x^k$.

Hence,

$$\lim_{k \rightarrow \infty} \|x^{i_k} - \tilde{y}^{i_k}\| = 0, \quad (15)$$

which imply that the sequences $(x^{i_k})_{k \in \mathbb{N}}$ and $(\tilde{y}^{i_k})_{k \in \mathbb{N}}$ have the same cluster points. From the definition of α_k in **Algorithm F**, \tilde{y}^k does not satisfy the inequality (10), that is, for all $v^k \in T(\tilde{y}^k)$ we have

$$\langle v^k, x^k - z^k \rangle < \delta \langle u^k, x^k - z^k \rangle. \quad (16)$$

As $\tilde{y}^{i_k} \rightarrow \tilde{x}$ we have by the continuity of T , that exist a sequence $v^{i_k} \in T(\tilde{y}^{i_k})$ such that, it is convergent to $\tilde{u} \in T(\tilde{x})$, taking this sequence and limits over the subsequence $(i_k)_{k \in \mathbb{N}}$ in (16) we have that $\langle \tilde{u}, \tilde{x} - \tilde{z} \rangle \leq \delta \langle \tilde{u}, \tilde{x} - \tilde{z} \rangle$. Then,

$$0 \geq (1 - \delta) \langle \tilde{u}, \tilde{x} - \tilde{z} \rangle = \frac{(1 - \delta)}{\tilde{\beta}} \langle \tilde{x} - (\tilde{x} - \tilde{\beta} \tilde{u}), \tilde{x} - \tilde{z} \rangle \geq \frac{(1 - \delta)}{\tilde{\beta}} \|\tilde{x} - \tilde{z}\|^2 \geq 0.$$

This means that $\tilde{x} = \tilde{z}$, the continuity of the projection and Proposition 2.5 implies $\tilde{x} \in S_*$.

We had proved that all cluster points belong to S_* . Now suppose that the sequence $(x^{n_k})_{k \in \mathbb{N}}$ is convergent to $\bar{x} \notin H(\bar{x}^{l_0}, \bar{u}^{l_0})$ for some $l_0 \in \mathbb{N}$, as $H(\bar{x}^{l_0}, \bar{u}^{l_0})$ is closed, and for all $n_k > l_0$ we have that $x^{n_k} \in H(\bar{x}^{l_0}, \bar{u}^{l_0})$ by Definition (11c) and (11b). This contradicts the fact that $\bar{x} \notin H(\bar{x}^{l_0}, \bar{u}^{l_0})$. The result holds. ■

Theorem 4.10 *The sequence generated by the algorithm converges to a point in the solution set S_* .*

Proof. By Proposition 4.5, the closure of the convex hull of \tilde{S}_* ($\bar{co}(\tilde{S}_*)$), is contained in $W(x^k)$ for all $k \in \mathbb{N}$ because $W(x^k)$ is convex and closed. Since $\bar{co}(\tilde{S}_*)$ is a nonempty, convex and closed set and $x^0 \notin \bar{co}(\tilde{S}_*)$, we can apply Lemma 2.8 with $S = \bar{co}(\tilde{S}_*)$ and $x = x^k$. Hence, we have that $(x^k)_{k \in \mathbb{N}} \subset B[\frac{x^0 + \bar{x}}{2}, \frac{\rho}{2}]$, where $\bar{x} = P_{\bar{co}(\tilde{S}_*)}(x^0)$ and $\rho = \|x^0 - \bar{x}\|$. All cluster points of the sequence belong to $\bar{co}(\tilde{S}_*)$ by Theorem 4.9. On the other hand, we have that $B[\frac{x^0 + \bar{x}}{2}, \frac{\rho}{2}] \cap \bar{co}(\tilde{S}_*) = \{\bar{x}\}$. This implies that $Cl(x^k)_{k \in \mathbb{N}} = \{\bar{x}\}$, therefore the sequence has only one cluster point and hence it converges to this cluster point \bar{x} . By Theorem 4.9, we conclude that $\bar{x} \in \tilde{S}_* \subseteq S_*$. ■

5 Numerical experiments

In this section we show some numerical experiments to test **Algorithm F** and compare it with [29, Algorithm 2.1]. We use MATLAB version R2015b on a PC with Intel(R) Core(TM) i5-4570 CPU 3.20GHz and Windows 7 Enterprise, Service Pack 1. For the calculation of the projection step we use the Quadratic Programming (quadprog) tool. In Examples 5.1 and 5.2 we use the stopping criterion $\|x^k - z^k\|^2 \leq 10^{-8}$, with x^k and z^k generated by the algorithm, $\delta = 0.01$, $\beta_k = 1$ for all $k \in \mathbb{N}$, $\theta = 0.5$. For “ x^0 ” we denote the initial point, “ $iter$ ” denotes the number of iteration of the algorithm, “ nT ” denotes the number of evaluations of the operator T . In Example 5.1 and 5.4 “ sol ” denotes the point at which the algorithm stops. In Example 5.3 we use $\theta = 0.25$ and tolerance $\|x^k - z^k\|^2 \leq 10^{-4}$. In Example 5.4 we use $\theta = \delta = 0.5$ and $\beta_k = 1$ for all $k \in \mathbb{N}$, the tolerance used was $\|x^k - z^k\|^2 \leq 10^{-80}$.

Example 5.1 [17, 29] Let $C = [0, 1] \times [0, 1]$ and $t = (x_1 + \sqrt{x_1^2 + 4x_2})/2$. We consider the Problem (1) with the operator $T : C \rightarrow \mathbb{R}^2$ defined as:

$$T(x_1, x_2) = (-t/(1+t), -1/(1+t)).$$

This example was introduced by Hadjisavvas and Schaible in [17] and also used in [29]. The operator T in this example is quasimonotone (i.e., for all $(x, u), (y, v) \in Gr(T)$ we have that $\langle u, y - x \rangle > 0$ implies $\langle v, y - x \rangle > 0$). The solution set is $S_* = S_0 = (1, 1)$. The results for this example are listed in **Table 1**.

Table 1. Results for example 5.1.						
	Alg F			Alg 2.1 in [29]		
x^0	iter(nT)	CPU time	sol	iter(nT)	CPU time	sol
(0,1)	1(3)	0.249602	(1,1)	3(2)	0.312002	(1,1)
(0,0)	1(3)	0.234001	(1,1)	50(406)	0.561604	(1,1)
(1,0)	2(4)	0.265202	(1,1)	71(331)	0.561604	(1,0.999)
(0.5,0.5)	0(2)	0.0156001	(1,1)	1(2)	0.234001	(1,1)
(0.2,0.7)	1(3)	0.249602	(1,1)	2(3)	0.280802	(1,1)
(0.1,0.7)	1(3)	0.249602	(1,1)	2(3)	0.296402	(1,1)

The following example with $n = 1$, $\rho(x) = \rho_1(x) = \|x\|^2$, and $a = 1$, is [29, Example 4.2].

Example 5.2 Let $C = [-a, a]^n$ with $a > 0$, and consider $T : C \rightarrow \mathbb{R}^n$ defined as $T(x) = (\rho_1(x), \rho_2(x), \dots, \rho_n(x))$ where, for all $i = 1, \dots, n$, $\rho_i : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is a continuous function satisfying $\rho_i(x) = 0$, iff, $x = 0$. Notice that $S_0 = -a(1, 1, \dots, 1)$ and $S_* = S_0 \cup (0, 0, \dots, 0)$. In this case, $S_0 \neq S_*$ and since T is continuous we can apply **Algorithm F** to find the solution. See the results for $\rho(x) = \rho_i(x) = \|x\|^2$ and $\rho(x) = \rho_i(x) = \|x\|$ for all $i = 1, \dots, n$, and $a = 1$, in the **Table 2**.

Table 2 Results for Example 5.2.						
			Alg F		Alg 2.1 in [29]	
ρ	n	x^0	iter(nT)	CPU time	iter(nT)	CPU time
$\ \cdot\ ^2$	1	0.1	88(178)	0.608404	512(2797)	1.79401
$\ \cdot\ ^2$	1	0.5	94(190)	0.592804	962(6428)	3.05762
$\ \cdot\ ^2$	1	-0.5	1(4)	0.249602	2(6)	0.296402
$\ \cdot\ $	5	$10^{-3}(1, \dots, 1)$	6(14)	0.280802	7(16)	0.327602
$\ \cdot\ $	50	$-10^{-1}(1, \dots, 1)$	1(4)	0.249602	2(6)	0.296402
$\ \cdot\ $	100	$-10^{-3}(1, \dots, 1)$	2(6)	0.265202	3(8)	0.296402

The following example is [29, Example 4.3].

Example 5.3 Consider the feasibility set $C = \{x \in \mathbb{R}^5 : x_i \geq 0, \ i = 1, 2, \dots, 5, \ \sum_{i=1}^5 x_i = a\}$ where $a > 0$. The problem,

$$\min F(x) \quad (17)$$

$$s.t. \ x \in C, \quad (18)$$

where $F(x) = \frac{\frac{1}{2}\langle Hx, x \rangle + \langle q, x \rangle + 1}{\sum_{i=1}^5 x_i}$ with H being a positive diagonal matrix with the same element h in the diagonal and $q = (-1, -1, \dots, -1)$, can be modeled as Problem 1, where T is a point-to-point operator defined by $T = \nabla F$. Note that $T(x) = \left(\frac{\partial F(x)}{\partial x_1}, \dots, \frac{\partial F(x)}{\partial x_5}\right)$, with $\frac{\partial F(x)}{\partial x_i} = \frac{hx_i \sum_{i=1}^5 x_i - \frac{1}{2}h \sum_{i=1}^5 x_i^2 - 1}{(\sum_{i=1}^5 x_i)^2}$. For this example, we have that $S_0 = \{\frac{1}{5}(a, \dots, a)\}$. Some values for δ are tested for better comparison with [29, Algorithm 2.1]. See the results in **Table 3**.

Table 3 Results for Example 5.3.						
			Alg F		Alg 2.1 in [29]	
x^0	δ	a	iter(nT)	CPU time	iter(nT)	CPU time
(0, 0, 5, 0, 0)	0.01	5	22(46)	0.218401	567(2269)	7.22285
(0, 2, 0, 2, 1)	0.01	5	36(74)	0.312002	509(2546)	6.81724
(0, 0, 5, 0, 0)	0.5	5	14(30)	0.156001	22(45)	0.249602
(0, 2, 0, 2, 1)	0.5	5	42(86)	0.374402	21(43)	0.218401
(1, 1, 1, 1, 6)	0.01	10	94(190)	0.639604	439(1757)	5.25723
(1, 1, 6, 1, 1)	0.01	10	101(204)	0.686404	482(1929)	6.00604
(1, 1, 1, 1, 6)	0.99	10	712(2138)	4.32123	40(81)	0.358802
(1, 1, 6, 1, 1)	0.99	10	846(2540)	5.22603	61(123)	0.514803

The following is an example with T be point-to-set and continuous. This example is inspired by [8, Example 2.5.8].

Example 5.4 Let the point-to-set operator $T : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$, be defined by

$$T(x, \theta) := \{t(\cos(\theta), \sin(\theta)) : t \geq x\},$$

and the set $C = \{(x, \theta) : x \geq 0, \ \theta \in [0, \pi/2]\}$. Consider Problem (1) for T and C .

It can be shown that the operator T is continuous but not *USC*. Since $(0, 0) \in T(0, \theta)$ for all $\theta \in [0, \pi/2]$ we have that the solution set $S_* = \{(0, \theta) : \theta \in [0, \pi/2]\}$. It can also be shown easily that $S_0 = \{(0, 0)\}$. In this example, we perform **Step 1** as follow. Given $x^k = (t_k, \theta_k)$, take $u^k = t_k(\cos(\theta_k), \sin(\theta_k))$. Our numerical results are reported in **Table 4** below.

Table 4 Results for Example 5.4.			
	Alg F		
x^0	iter(nT)	CPU time	sol
$(1, \pi/2)$	7(16)	0.296402	$(0, 0)$
$(0.5, \pi/3)$	145(292)	1.09201	$(0, 0)$
$(0.1, \pi/2)$	378(758)	2.77682	$(0, 0)$
$(100, \pi/2)$	6(15)	0.327602	$(0, 0)$
$(0.1, \pi/10)$	89(180)	0.702005	$(0, 0)$
$(1, \pi/100)$	7(16)	0.312002	$(0, 0)$
$(20, \pi/6)$	3(8)	0.280802	$(0, 1.7526 * 10^{-16})$
$(10, \pi/4)$	3(8)	0.312002	$(0, 8.4431 * 10^{-12})$
$(1500, \pi/8)$	5(12)	0.296402	$(0, 4.3692 * 10^{-9})$

Remark 5.5 The numerical results indicate that the performance of our algorithm is comparable to that of similar methods in the literature, e.g., Algorithm 2.1 from [29]. In Example 5.1 and Example 5.2 we can observe a slight advantage of our algorithm for some choices of the initial point. In Example 5.3 we note that some choices of δ give us different behavior. Namely, when δ is close to 0, our algorithm requires a smaller number of iterations and less CPU time. This situation is reversed when δ is close to 1. Indeed, for this case [29, Algorithm 2.1] requires fewer iterations and less CPU time than ours. That is confirmed by the fact that for $\delta = 0.5$ both algorithms have a similar performance. In Example 5.4 the implementation is possible because for all $x \in \text{dom}(T)$, the set $T(x)$ is a ray, then the computational implementation of the **Linesearch F** is possible because the optimization problem

$$\max \langle y, w \rangle \text{ such that } y \in T(x),$$

is implementable.

6 Conclusions

We present an algorithm for solving the Variational Inequality Problem in finite dimensional Euclidian spaces for point-to-set operators. We establish convergence without any monotonicity assumption. Our numerical experiments show that when the operator T is point-to-point, our algorithm has a competitive performance with respect to a similar algorithm in the literature. The **Linesearch F** requires the knowledge of the whole set $T(\alpha z + (1 - \alpha)x)$. Indeed, it requires to verify that $\forall u_\alpha \in T(\alpha z + (1 - \alpha)x)$, the inequality $\langle u_\alpha, x - z \rangle < \delta \langle u, x - z \rangle$ holds. The question of finding an implementable linesearch for the point-to-set case is an open problem and the subject of our future research.

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